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Stiffness and Damage Identification with Model Reduction Technique

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ABSTRACT

For structural health monitoring, it is impractical to use full measurement to identify the unknown parameters. When one attempts to retrieve second-order parameters from the identified state space model, various methodologies impose different restrictions on the number of sensors and actuators employed, assuming that all the modes of the structure can be successfully identified. This paper explores the possibility of performing this inverse vibration problem without minimal restriction on the number of the measurements. Based on the numerical analysis of measured response due to known excitation, structural parameters such as stiffness values are identified and compared with the intended design values for damage detection. The proposed methodology is based on the Eigensystem Realization Algorithm (ERA) and the Observer/Kalman Filter Identification (OKID) approach to identify a first-order state space model. The first-order model is then transformed into a second-order model in order to evaluate the system stiffness. The focus is on estimation of all stiffness values from the condensed stiffness matrices by model reduction making use of System Equivalent Reduction Expansion Process (SEREP). The required optimization is accomplished by the Genetic Algorithm (GA). The efficiency of the proposed technique is shown by numerical examples for multi-storey shear building subjected to random excitation, taking into consideration the effects of noisy data. The results indicate that the proposed methodology offers reasonably accurate identification in terms of locating and quantifying damage.

INTRODUCTION

System identification methods have been extensively studied in recent years for structural damage assessment. The methods considered in this paper are based on two

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steps. In the first step, identification of modal parameters (first-order) is addressed by OKID and ERA. Once the modal parameters have been identified, structural parameters (second-order) determined in the second step. The latter is challenging due to practical issues such as limited sensor information.

When one tries to retrieve second-order parameters from first-order parameters of the identified state space model, various methodologies impose different restrictions on the number of sensors and actuators employed. For example, some methods require as many sensors and actuators as the number of identified modes [1]. Others require that the number of sensors should be equal to the number of identified modes, with one co-located sensor-actuator pair [2]. Tseng *et al*. presented a further generalization case where the number of actuators is equal to the number of second-order modes [3,4]. Lus utilized mixed type information, thereby enabling one to treat information from a sensor or an actuator in an analogous fashion [5]. The requirements are that all the degrees of freedom (DOFs) should contain either a sensor or an actuator. Furthermore, there should be at least one co-located sensor-actuator pair. This requirement is usually not practical in engineering application.

In practice, the number of sensors for measurement is often limited making the task of identifying systems with many unknown parameters difficult, particularly when one attempts to identify the full system in one go. Alternatively, a reduced, or condensed, system is identified corresponding to the number of sensors used. This, however, does not necessarily give information of all unknown parameters. To this end, the objective of this study is to determine individual stiffness parameters of the full system from the identified condensed system. In this respect, SEREP to reduce the structural stiffness matrix is referred to [6]. On the basis of condensation, all stiffness parameters in the entire system can be recovered by extracting sufficient information with fixed sensors. The required optimization is accomplished by a versatile search method of GA. The study is carried out herein by means of numerical simulation including effects of input and output (I/O) noise as well as effects of number of sensors.

BASIC FORMULATION

The dynamic response of a *N*-DOF linear structural system can be represented by

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{L}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{1}$$

where $\mathbf{M}$, $\mathbf{L}$ and $\mathbf{K}$ are the mass, damping, and stiffness matrices of the structure, respectively, $\mathbf{q}$ is a displacement vector and the overdot denotes differentiation with respect to time t. The equations of motion and the measurement equations can be written in the first-order state space form as

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \tag{2}$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) \tag{3}$$

where $\mathbf{x}(k)$ is a $n \times 1$ state vector, $\mathbf{y}(k)$ a $m \times 1$ observation vector, and $\mathbf{u}(k)$ a $r \times 1$ input vector.

OKID and ERA

For zero initial conditions, Eqs. (2) and (3) can also be written in matrix form for a sequence of 'l' consecutive time steps as

$$\underset{m\times l}{\mathbf{y}} = \underset{m\times[(r+m)p+r]}{\overline{\mathbf{Y}}} \quad \underset{[(r+m)p+r]\times l}{\mathbf{V}} \tag{4}$$

where $\mathbf{y}$ is an output data matrix, $\overline{\mathbf{Y}}$ contains all observer Markov parameters to be determined and $\mathbf{V}$ is a input matrix. Having identified the observer Markov parameters, the system's Markov parameters can be retrieved using the recursive formula [7]. ERA begins by forming the generalized Hankel matrix, composed of the Markov parameters. The ERA process starts with the factorization of the first Hankel matrix using singular value decomposition, $\mathbf{H}(0) = \mathbf{R}\boldsymbol{\Sigma}\mathbf{S}^T$. This is the basic formulation of realization for the ERA [8]. The triplet

$$\mathbf{A} = \boldsymbol{\Sigma}_\mathbf{n}^{-1/2}\mathbf{R}_\mathbf{n}^\mathbf{T}\mathbf{H}(1)\mathbf{S}_\mathbf{n}\boldsymbol{\Sigma}_\mathbf{n}^{-1/2},\ \mathbf{B} = \boldsymbol{\Sigma}_\mathbf{n}^{1/2}\mathbf{S}_\mathbf{n}^\mathbf{T}\mathbf{E}_\mathbf{r},\ \mathbf{C} = \mathbf{E}_\mathbf{m}^\mathbf{T}\mathbf{R}_\mathbf{n}\boldsymbol{\Sigma}_\mathbf{n}^{1/2} \tag{5}$$

is a minimum realization where $\mathbf{E}_r = [\mathbf{I}_{r\times r} \quad \mathbf{0}_{r\times r} \quad \mathbf{0}_{r\times r} \quad \cdots \quad \mathbf{0}_{r\times r}]^T$ with $\mathbf{I}$ denoting an identity matrix and $\mathbf{0}$ denoting a matrix whose elements are all zeros, and $\mathbf{E}_m$ is defined similarly.

Conversion from First-order Model to Second-order Model

The first-order model in Eq. (2) is written based on the second-order model in Eq. (1) and hence their eigenvectors are related through similarity transformation. The eigenvalues and eigenvectors of the first-order system can first be computed using $\mathbf{A}$ and then transformed and properly scaled to become the eigenvectors $\mathbf{P}$ and eigenvalues $\boldsymbol{\Gamma}$ of the structural model in Eq. (1). The mass, damping, and stiffness matrices can be obtained by… [5]:

$$\mathbf{M} = (\mathbf{P}\boldsymbol{\Gamma}\mathbf{P}^\mathbf{T})^{-1}, \quad \mathbf{L} = -\mathbf{M}\mathbf{P}\boldsymbol{\Gamma}^2\mathbf{P}^\mathbf{T}\mathbf{M}, \quad \mathbf{K} = -(\mathbf{P}\boldsymbol{\Gamma}^{-1}\mathbf{P}^\mathbf{T})^{-1} \tag{6}$$

INCOMPLETE MEASUREMENT WITH MODEL REDUCTION TECHNIQUE

The previous section shows that each DOF must contain either an actuator or a sensor, with at least one DOF containing a co-located sensor-actuator pair in order to identify mass, damping and stiffness matrices. This stringent requirement is often impractical in real life applications. Therefore, a new methodology is presented for identification of stiffness matrices without meeting the stringent requirement. The model reduction technique will make use of the System Equivalent Reduction Expansion Process (SEREP) as proposed by O'Callahan *et al.* [6]. Suppose the total DOFs of the full model are divided into primary DOFs (p), which will be retained in the reduced models, and the secondary DOFs (s), which will be excluded from the full model. Accordingly, matrix **M** and **K** can be partitioned as

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{pp} & \mathbf{M}_{ps} \\ \mathbf{M}_{sp} & \mathbf{M}_{ss} \end{bmatrix}_{N\times N} \qquad \mathbf{K} = \begin{bmatrix} \mathbf{K}_{pp} & \mathbf{K}_{ps} \\ \mathbf{K}_{sp} & \mathbf{K}_{ss} \end{bmatrix}_{N\times N} \tag{7}$$

Let N be the number of DOFs and n the number of sensors used. Using the reduced model, the $n \times n$ mass and stiffness matrices can be obtained using OKID and ERA. An eigen-analysis on these reduced matrices yields the identified natural frequencies and mode shapes (termed the identified values), denoted as

$$\mathbf{\Lambda}_{n\times n} = diag(\omega_1^2 \cdots \omega_n^2) \quad \mathbf{\Phi}_{n\times n} = [\boldsymbol{\varphi}_{p1} \cdots \boldsymbol{\varphi}_{pn}] \tag{8}$$

The modal matrix $\mathbf{\Phi}$ in Eq. (8) is then expanded to include the secondary DOFs using

$$\boldsymbol{\varphi}_{si} = -(\mathbf{K}_{ss} - \omega_i^2 \mathbf{M}_{ss})^{-1}(\mathbf{K}_{sp} - \omega_i^2 \mathbf{M}_{sp})\boldsymbol{\varphi}_{pi}, \quad i = 1, \cdots, n \tag{9}$$

The expanded, mass normalized modal matrix is then finally computed from

$$\mathbf{\Theta} = \mathbf{\Phi}(\overline{\mathbf{\Phi}}^T \mathbf{M} \overline{\mathbf{\Phi}})^{-1/2} \equiv \begin{bmatrix} \mathbf{\Theta}_p \\ \mathbf{\Theta}_s \end{bmatrix} \tag{10}$$

where $\overline{\mathbf{\Phi}}$ is the modal matrix formed from the n normalized mode shapes. The SEREP transformation now becomes

$$\mathbf{P}_{SEREP} = \begin{bmatrix} \mathbf{\Theta}_p \\ \mathbf{\Theta}_s \end{bmatrix} \mathbf{\Theta}_p^{-1} \tag{11}$$

Note that the eigenvalues and eigenvectors obtained from the stiffness and mass matrices of the reduced model refined by SEREP method (for simplicity, termed as the analytically reduced values) should theoretically be identical to Eq. (8). Hence the procedure to obtain **K** and **M** is to assume an initial set of values for $\mathbf{K}_{sp}$, $\mathbf{K}_{ss}$, $\mathbf{M}_{sp}$ and $\mathbf{M}_{ss}$ to be used with Eq. (9) to be able to compute Eq. (11). The latter is then used to compute the eiqenvalues and eigenvectors and check whether they match Eq. (8).

On the basis of SEREP, all stiffness parameters in the entire system can be recovered by extracting sufficient information using fixed location sensors as proposed below. The computational procedure is as follows.

a) Identify the reduced stiffness matrix ($n \times n$) and mass matrix ($n \times n$) by OKID/ERA with conversion from first-order model to second-order model based on data from n sensors in the N-DOF structure. The n eigenvalues and eigenvectors can then be computed. Apply SEREP to obtain the analytical reduced stiffness and mass matrix and solved the corresponding n eigenvalues and eigenvectors.

b) Repeat Step (a) as many times as possible by ignoring one sensor at a time. If necessary, repeat by ignoring more sensors until sufficient information is obtained. It can be shown that the number of ways to ignore sensors is ${}^{n}C_{n-m}$, where m is the number of sensor ignored.

c) Solve the fitness function as described below. Note that it is possible to obtain more information than required (i.e. *N*). This over-determined system can be solved using least square method or, in this study, by GA.

The procedure for recovering the storey stiffness values from the reduced stiffness matrices is highly nonlinear and GA is used in this study. Eigenvalue and eigenvector from the identified and analytically reduced matrices are used in the fitness function. To estimate the accuracy of the eigenvectors, a correlated coefficient for modal vector (CCFMV) value is defined as [9]

$$CCFMV(\boldsymbol{\varphi}_j^c, \boldsymbol{\varphi}_j^I) = \frac{\left|\boldsymbol{\varphi}^{T}{}_j^c \cdot \boldsymbol{\varphi}_j^I\right|}{[(\boldsymbol{\varphi}^{T}{}_j^c \cdot \boldsymbol{\varphi}_j^c)(\boldsymbol{\varphi}^{T}{}_j^I \cdot \boldsymbol{\varphi}_j^I)]^{1/2}} \tag{12}$$

where $\boldsymbol{\varphi}_j^c$ and $\boldsymbol{\varphi}_j^I$ are the exact and the identified of the j^{th} eigenvector, respectively. A CCFMV close to 1 suggests that the two modes or vectors are well correlated and a value close to 0 indicates uncorrelated modes. Therefore, the fitness function, in terms of the differences not only between the eigenvalue, but also between the mode shapes in CCFMV, is as follow

$$\varepsilon = \sum_{j=1}^{n} [(1 - \frac{\lambda_j^c}{\lambda_j^I})^2 + (1 - CCFMV(\boldsymbol{\varphi}_j^c, \boldsymbol{\varphi}_j^I))^2] \tag{13}$$

The problem of recovering the stiffness values is transformed into minimization of Eq. (13).

NUMERICAL RESULTS AND DISCUSSION

To validate the proposed damage identification approach, numerical simulation study is carried out to identify the location and extent of structural damage in a twelve-storey shear building. The input is a random excitation whereas the response measurements are numerically simulated accelerations.

Identification of the Stiffness from Reduced Stiffness Matrix

The undamaged case is first considered here using incomplete measurements. In reality, I/O noise is inevitable and plays an important role in the accuracy and efficiency of the damage identification results. Obviously with increasing noise level, the leakage phenomenon becomes more severe and generally gives worse identification results. The storey stiffness values are far more difficult to identify accurately when the number of sensors decreases. The more measurements (presumably with good accuracy), the better identification results. More sensors provide more information about the system, which gives better identification result. For this reason, the identification accuracy can be improved by increasing the number of sensors used. Tables I and II illustrate the effects of I/O noise and effect of number of sensors for structural parameter estimation problem. Both the I/O signals are polluted with 10% root-mean-square Gaussian white noise in order to simulate measurement noise. As shown in Tables I and II, the maximum absolute error is 7.7% for the case of using 6 sensors and, 4.1% for the case of 8 sensors with no noise. However, with 10% noise, the maximum errors increase to 10.2% and 8.9% for the cases of 6 and 8 sensors, respectively. In the case of 8 sensors, the stiffness obtained is much better than the case of 6 sensors. The mean error is reduced from 4.3% for 6 sensors to 2.2% for 8 sensors for no noise data and from 6.3% for 6 sensors to 5.2% for 8 sensors for noisy data (10% noise).

TABLE I. IDENTIFIED STOREY STIFFNESS FOR UNDAMAGED CASE UNDER NO NOISE

Storey	Exact Stiffness (kN/m)	Identified Stiffness in kN/m (Error in Bracket)	
		6 sensors	8 sensors
1	500	461.4 (-7.72%)	496.1 (-0.78%)
2	500	512.3 (2.46%)	480.6 (-3.88%)
3	500	518.3 (3.66%)	492.7 (-1.46%)
4	500	522.3 (4.46%)	490.6 (-1.88%)
5	500	478.9 (-4.22%)	485.5 (-2.90%)
6	500	481.2 (-3.76%)	490.7 (-1.86%)
7	500	523.1 (4.62%)	491.2 (-1.76%)
8	500	488.9 (-2.22%)	499.0 (-0.20%)
9	500	465.1 (-6.98%)	510.4 (2.08%)
10	500	523.1 (4.62%)	506.7 (1.34%)
11	500	514.9 (2.98%)	518.0 (3.60%)
12	500	521.2 (4.24%)	520.4 (4.08%)
Mean Error		4.33%	2.15%
Max. Error		7.72%	4.08%

TABLE II. IDENTIFIED STOREY STIFFNESS FOR UNDAMAGED CASE UNDER 10% NOISE

Storey	Exact Stiffness (kN/m)	Identified Stiffness in kN/m (Error in Bracket)	
		6 sensors	8 sensors
1	500	467.8 (-6.44%)	511.2 (2.24%)
2	500	477.0 (-4.60%)	544.5 (8.90%)
3	500	534.2 (6.84%)	488.2 (-2.36%)
4	500	449.0 (-10.2%)	523.1 (4.62%)
5	500	471.4 (-5.72%)	542.1 (8.42%)
6	500	477.9 (-4.42%)	476.7 (-4.66%)
7	500	521.7 (4.34%)	523.8 (4.76%)
8	500	523.5 (4.70%)	517.8 (3.56%)
9	500	477.9 (-4.42%)	534.1 (6.82%)
10	500	456.7 (-8.66%)	533.4 (6.68%)
11	500	457.3 (-8.54%)	467.1 (-6.58%)
12	500	533.5 (6.70%)	488.1 (-2.38%)
Mean Error		6.30%	5.17%
Max. Error		10.20%	8.90%

Damage Detection

Damage in vertical supporting members (such as columns) can be reflected by reduction in storey stiffness value. Two damage scenarios are studied: (1) with single damage and (2) with multiple damages. Scenario 1 contains 30% damage in the fourth storey (i.e. the remaining stiffness is 70% of the original value). Scenario 2 has two damage locations: 20% damage in the second storey and 40% in the fifth storey. It is possible to detect damage by identifying storey stiffness values and comparing them with the corresponding values of the original (presumably undamaged) state. The stiffness integrity index is 1 for no loss in stiffness (no damage) and 0 for complete loss of stiffness (complete damage). Figures 1 and 2 present the identified stiffness integrity index for Scenario 1 and Scenario 2 with 10% noise. The maximum error in the identified stiffness integrity index is 8.3% for 6 sensors and 6.4% for 8 sensors in Scenario 1 under 10% I/O noise. The maximum error in Scenario 2 is larger – 13.0% and 8.6% for 6 sensors and 8 sensors, respectively, under 10% noise. The proposed approach is thus effective in identifying the damage locations and extents with good engineering accuracy, considering the adverse influence of noise and incomplete measurement information. It can be shown that the stiffness integrity indices are identified with accuracy and this should be considered as an alternative in locating and quantifying damage in a linear dynamic structural system. This is because SEREP preserves the natural frequencies and mode shapes during the reduction process. These results are better than those in which static condensation is used and this actually introduces modeling error when applied to structural dynamic problems [10].

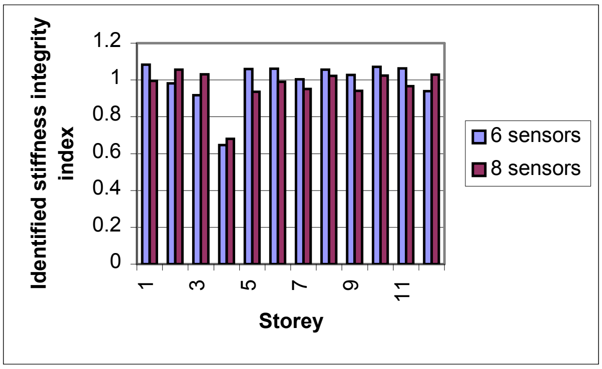

Figure 1. Identified stiffness integrity index of each storey for Scenario 1 under 10% noise.

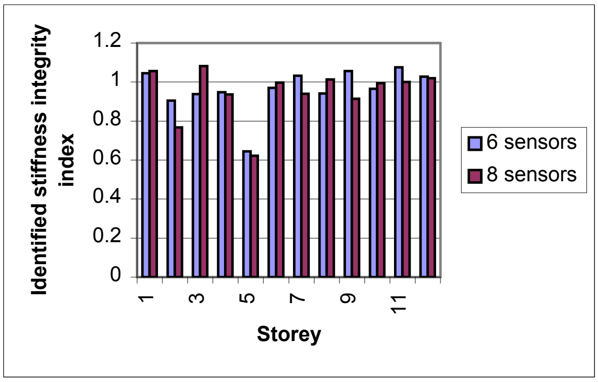

Figure 2. Identified stiffness integrity index of each storey for Scenario 2 under 10% noise.

CONCLUSIONS

This study focuses on development of a practical numerical identification strategy for off-line applications to linear structures. A new methodology for identification of stiffness and damage using incomplete measurements is presented. The main research significance is that with the model reduction technique, it is possible to utilize several reduced stiffness matrices to enable one to find the stiffness and stiffness integrity index of each floor. It has been shown that condensation method can be used to recover the full set of stiffness parameters from the identified condensed stiffness matrices. Several factors of practical consideration including number of sensors and I/O noise are studied. Obviously with increasing noise level, the leakage phenomenon becomes more severe and generally gives worse identification results. The proposed approach is effective in identifying the damage locations and extents with SEREP. This is because static condensation introduces modeling errors when applied to structural dynamics problems, as inertia forces are not accounted for in the condensation process. This approach overcomes this problem as well as the necessity of having either a full set of sensors or a full set of actuators, thereby allowing fewer number of sensor and actuator than those required in previously discussed approaches.

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